**Problem 1. Potential games (5 points)**

a) Consider the two-player potential game, where the row player with actions: $U$, $D$ (for up and down rows) aims to minimize the cost function $A \in \mathbb{R}^{2\times 2}$ and the column player with actions: $L, R$ (for left and right columns) aims to minimize the cost $B \in \mathbb{R}^{2\times 2}$, with $A = B = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$.

  i) What is(are) the potential function minimizer(s)? (.5 point)

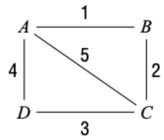
  ii) What is(are) the Nash equilibrium(equilibria) of the game? (2 points)

  iii) Is there an admissible Nash equilibrium? If so, provide it. Else, justify. (.5 point)

b) Consider a zero-sum game with the cost function of the minimizer (row player) being $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that the game has an exact potential function if and only if $a + d = b + c$. (2 points)

**Problem 2. Best-response and congestion games (6 points)**

Consider the following road network that connects the four cities $A$, $B$, $C$, $D$ through five roads. There are 10 vehicles and they have two types. Type 1: 6 vehicles with origin $A$ and destination $D$; Type 2: 4 vehicles with origin $B$ and destination $D$. Let $C(n) > 0$ denote the cost of traveling road $r \in \{1, 2, 3, 4, 5\}$, where $n$ is the number of vehicles using a given road and $C : \mathbb{N} \to \mathbb{R}$ is monotonically increasing. So, the action space for Type 1 players is $\Gamma := \{\{4\}, \{5, 3\}, \{1, 2, 3\}\}$.




a) What is the action space, $\Sigma$, for Type 2 players? (1 point)

From now on, consider the initial strategies where vehicles in each type distribute equally to their respective feasible action spaces. So, from Type 1 vehicles, 2 choose each of the three strategies in $\Gamma$.

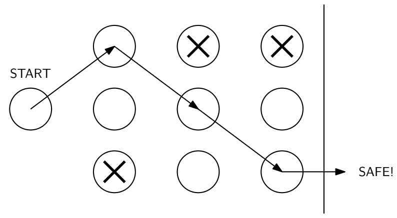
b) What is the cost of a player choosing $\{1, 2, 3\}$? (2 points)

c) Consider a player who chose $\{1, 2, 3\}$. What is this player's best response strategy? (1 point)

d) Justify that this game has a Nash equilibrium strategy. (1 point)

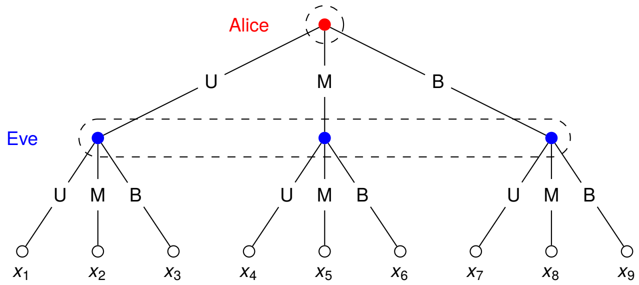
e) What is an upper bound on the number of iterations for which the best-response dynamic in the above game will converge to a Nash equilibrium? (1 point)

**Problem 3. Multi-stage game (9 points)**




Player 1 (Alice) is trying to escape, going from the **start node** to the **safe zone** without being intercepted. At every stage of the game, Alice moves one step closer to the safe zone. She can decide to continue on the same row, or instead move diagonally one row up or one row down. Player 2 (Eve) is trying to stop Alice. At each stage, Eve is aware of Alice's current position, and she is allowed to block one of the three rows in the next stage, taking her decision simultaneously with Alice. If she selects the row corresponding to Alice's next move, she stops her and she wins the game getting a cost of $-1$, while Alice gets a cost of 1.

Let U, M, B denote the choice of Up, Middle and Bottom rows, respectively. First, consider the case with 3 rows and only 1 stage (that is, only 1 chance for Eve to stop Alice). The game tree for Alice is shown here.

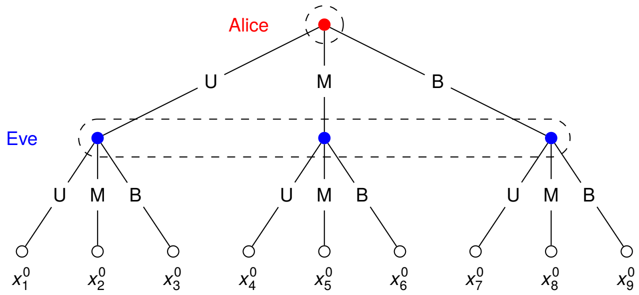



a) Fill in the $x_i$'s above. Formulate the game in matrix form. (1 point)

b) Show that $y^* = z^* = (1/3, 1/3, 1/3)$ is the Nash equilibrium of the game. (1 point)

c) Verify that the value of the game is $-1/3$. (1 point)

Now, consider the case where there are 3 rows and 2 stages (that is, 2 chances for Eve to stop Alice). We aim to use backward induction to determine the value of the game and fill in the entries denoted as $x_i$'s below, corresponding to Alice's costs.




d) First, if Alice and Eve choose the same move in the first stage, clearly Eve catches Alice and the game ends. Based on this, fill in $x_1^0, x_5^0, x_9^0$ (.5 point).

e) Next, note that if Alice chooses M in the first stage and Eve does not choose M, the game at the next stage will be identical to the one discussed in the previous part. Based on this observation, fill in $x_4^0, x_6^0$. (1 point)

f) Now, consider Alice choosing $U$ and Eve choosing $M$ or $B$, in the first stage. In the next stage, Alice can choose either $U$ or $M$. Draw the matrix game for this next stage (1 point); Verify that $y^* = [1/2, 1/2]$, and $z^* = [1/2, 1/2, 0]$ is a Nash equilibrium. Based on this fill in $x_2^0, x_3^0$. (1 point)

g) By symmetry, fill in $x_6^0, x_7^0$. (.5 point)

h) Formulate the game with the game tree shown above in matrix form (1 point).

i) Verify that the Nash equilibrium strategy of Alice in her first move is $y^* = [4/11, 3/11, 4/11]$. (1 point)